# Canonical quantization:

Take 3-manifold $M_3$ to be of the form

$$M_3 \cong \Sigma \times \mathbb{R}$$

Riemann surface

Canonical quantization $\longrightarrow$ Hilbert space $\mathcal{H}_\Sigma$

Recall:

$$CS(A) = \frac{k}{8\pi^2} \int_{M_3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Taking $M_3 = \Sigma \times \mathbb{R}$ and choosing the gauge $A_0 = 0$, gives

$$\mathcal{L} = \frac{k}{8\pi^2} \int dt \int_\Sigma \varepsilon^{ij}\, \mathrm{Tr}\, A_i \frac{d}{dt} A_j .$$

$\longrightarrow$ Poisson brackets:

$$\{A_i^a(x), A_j^b(y)\} = \frac{4\pi^2}{k} \cdot \varepsilon_{ij}\, \delta^{ab}\, \delta^2(x-y)$$

Due to gauge choice, system is subject to:

$$\frac{\delta \mathcal{L}}{\delta A_0} = 0 \iff \varepsilon^{ij} F_{ij}^a = 0 \qquad (*)$$

We quantize the system by first constraining the phase space $\longrightarrow$ symplecting quotient:

recall $\mathcal{A}_\Sigma \cong \Omega^1(\Sigma, \mathfrak{g})$ symplectic manifold with non-degenerate symp. form

$$\omega(\alpha, \beta) = -\frac{K}{8\pi^2} \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta), \quad \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g})$$

Then $\mathcal{G}_\Sigma \cong \mathrm{Map}(\Sigma, G)$ acts on $\mathcal{A}_\Sigma$ by gauge transformations

Imposing the constraint $(*)$ is then equivalent to defining the quotient space:

$$\mathcal{M}_\Sigma = F^{-1}(0) / \mathcal{G}_\Sigma$$

"symplectic quotient"

The symplectic structure on $\mathcal{M}_\Sigma$ is given by

$$\omega_{\tilde{A}}(\tilde{\alpha}, \tilde{\beta}) = \omega_A(\alpha, \beta) \quad \text{for } \tilde{\alpha}, \tilde{\beta} \in T_A \mathcal{M}_\Sigma$$

$\rightarrow$ definition is independent of choice of $A$, $\alpha$, $\beta$

## Holomorphic viewpoint:

Lift action of gauge group $\mathcal{G}_\Sigma$ to $L_\Sigma$.

$\longrightarrow$ complex line bundle $\mathcal{L}$ on $\mathcal{M}_\Sigma$ with connection $\nabla$ of curvature $-i\omega$

To quantize $\mathcal{M}_\Sigma$, we need to pick hol. sections of $\mathcal{L}$ $\longrightarrow$ choice of complex structure

1) pick a complex structure $J$ on $\Sigma$

2) for $\alpha \in T\mathcal{A}_\Sigma$, define

$$J_{\mathcal{A}_\Sigma} \alpha = -J\alpha$$

3) $T\mathcal{A}_\Sigma = T^{(1,0)}\mathcal{A}_\Sigma \oplus T^{(0,1)}\mathcal{A}_\Sigma$, where $T^{(1,0)}\mathcal{A}$ and $T^{(0,1)}\mathcal{A}$ consist of $(0,1)$-forms and $(1,0)$-forms on $\Sigma$ with values in $\mathrm{ad}(Q)$.

4) $\omega$ is of type $(1,1)$.

5) the connection $d_A$ decomposes as

$$d_A = \partial_A + \bar{\partial}_A,$$

6) complexified $\mathcal{G}_{\mathbb{C}}$-action is determined by:

$$\bar\partial_A \longmapsto g\cdot\bar\partial_A\cdot g^{-1}, \quad \partial_A \longmapsto \bar g\cdot\partial_A\cdot\bar g^{-1}$$

7) The set $\mathcal{A}/\mathcal{G}_{\mathbb{C}}$ can be identified with the set $\mathcal{M}_J$ of equivalence classes of hol. structures on $Q_{\mathbb{C}}$.

8) $\mathcal{T} \equiv$ moduli space of complex structures on $\Sigma$

$\rightarrow$ $t\in\mathcal{T}$ determines $J_t$ and $\mathcal{M}_{J_t}$

$\rightarrow$ $\mathcal{M}\times\mathcal{T}$ can be regarded as a bundle over $\mathcal{T}$ with fibers varying with $t$.

9) Introduce quantum bundle $\mathcal{H}_Q$ over $\mathcal{T}$. The fiber $\mathcal{H}_Q|_t$ over a point $t$ is $H^0(\mathcal{M}_{J_t}, \mathcal{L})$ (holomorphic sections)

10) There is a "projectively flat conn." $\nabla$ on $\mathcal{H}_Q \rightarrow \mathcal{T}$.

# Construction of the connection

First, we need to describe the connetion $\nabla$ on $L_\Sigma$, with curvature $-i\omega$.

Define

$$\nabla_u \Psi(A) = \int_\Sigma d^2z\, u^a_z \frac{D}{DA^a_z(z)} \Psi(A),$$

$$\nabla_{\bar u} \Psi(A) = \int_\Sigma d^2z\, \bar u^a_{\bar z} \frac{D}{DA^a_{\bar z}(z)} \Psi(A),$$

for $u$ and $\bar u$ adjoint valued $(1,0)$ and $(0,1)$ forms on $\Sigma$ and $d^2z = i dz\, d\bar z$.

We need

$$\left[ \frac{D}{DA^a_w(w)}, \frac{D}{DA^b_{\bar z}(z)} \right] = \overbrace{-i \frac{K}{4\pi} \delta_{ab}\, \delta_{z\bar w}(z,w)}^{=2\pi i\omega}$$

$$\left[ \frac{D}{DA^a_z(z)}, \frac{D}{DA_w{}^b(w)} \right] = \left[ \frac{D}{DA_{\bar z}{}^a(z)}, \frac{D}{DA_{\bar w}{}^b(w)} \right] = 0$$

$\rightarrow$ take:

$$\frac{D}{DA_z} = \frac{\delta}{\delta A_z} - \frac{K}{8\pi} A_{\bar z}$$

$$\frac{D}{DA_{\bar z}} = \frac{\delta}{\delta A_{\bar z}} + \frac{K}{8\pi} A_z$$

→ holomorphic sections have to satisfy:

$$\frac{D}{DA_z} \Psi = 0 \quad , \quad \Psi \in H^0(\mathcal{A}_\Sigma, L_\Sigma) =: \mathcal{H}_Q|_J$$

To study the objects $\widetilde{\mathcal{H}}_Q|_J = H^0(\mathcal{M}_J, \mathcal{L})$, we need to further restrict to $\mathcal{G}_{\mathbb{C}}$-inv. subspace of $\mathcal{H}_Q|_J$.

→ the condition for $\mathcal{G}_{\mathbb{C}}$-invariance is

$$\left( -D_{\bar z} \frac{D}{DA_{\bar z}^{a}(z)} + \frac{K}{4\pi} \delta_{ab} F^{b}_{\bar z z}(z) \right) \Psi = 0$$

where $F^{a}_{\bar z z}(z)$ is the curvature of the connection $A$, and $D_{\bar z} = \frac{\partial}{\partial \bar z} + A_{\bar z}$.

We will now construct the connection on

$$\begin{tikzcd} \widetilde{\mathcal{H}}_Q|_J \arrow[r] & V \arrow[d] \\ & \mathcal{T} \end{tikzcd}$$

Note that the space $\mathcal{A}_\Sigma$ is infinite-dim., while $\mathcal{M}_\Sigma$ has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}_\Sigma = (g-1) \cdot \dim G$$

This comes from the fact that flat connections are characterized by "Wilson lines" (holonomies around cycles of $\Sigma$)

$\mathcal{T}$ is a complex manifold

$\longrightarrow$ write $\delta^{(1,0)}$ and $\delta^{(0,1)}$ respectively for the $\partial$ and $\bar{\partial}$ operators on $\mathcal{T}$.

Explicitly:

$$\delta^{(1,0)} = \int_\Sigma \delta t_{\bar{z}\bar{z}} \frac{\delta}{\delta t_{\bar{z}\bar{z}}}$$

$$\delta^{(0,1)} = \int_\Sigma \delta t_{zz} \frac{\delta}{\delta t_{zz}}$$

Working "upstairs" on $\mathcal{A}$, the projectively flat connection on $\mathcal{H}_Q$ is

$$\delta^{\mathcal{H}_Q} = \delta^{(1,0)} - \frac{it}{4} \cdot \frac{4\pi}{K} \int_\Sigma \delta J_{\bar{z}}^{z} \frac{D}{DA_z^a(z)} \frac{D}{DA_{\bar{z}}^a(z)}.$$